

Γ -Conformal Algebras

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Abstract

Γ -conformal algebra is an axiomatic description of the operator product expansion of chiral fields with simple poles at finitely many points. We classify these algebras and their representations in terms of Lie algebras and their representations with an action of the group Γ . To every Γ -conformal algebra and a character of Γ we associate a Lie algebra generated by fields with the OPE with simple poles. Examples include twisted affine Kac-Moody algebras, the sin algebra (which is a “ Γ -conformal” analogue of the general linear algebra) and its analogues, the algebra of pseudodifferential operators on the circle, etc.

0 Introduction

In the past years there has been a number of papers, where the effect of splitting of a pole of order N in the operator product expansion (OPE) occurs. Examples include representations of twisted affine Kac-Moody algebras [K-K-L-W], of the Lie algebras of Quantum torus [G-K-L], of Quantum affine algebras [F-J], and of the central extension of the double of Yangian [Kh-L-P].

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These constructions found many applications in the theory of integrable systems and exactly solvable models in quantum field theory.

The simplest motivation to consider the OPE with simple poles, but in the shifted points is the following. Consider the OPE of two local fields $a(z)$ and $b(z)$:

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} \quad (0.1)$$

and replace $(z-w)^n$ by its q -analogue $(z-w)_q^n = (z-w)(z-qw) \cdots (z-q^{n-1}w)$. The result of this q -deformation is an OPE with only simple poles. A natural generalization is to consider Γ -deformation, where Γ is a subgroup of the multiplication group of non-zero complex numbers \mathbb{C}^\times . Namely, for an n -element subset S of Γ we let

$$(z-w)_S^n = \prod_{\alpha \in S} (z-\alpha w)$$

and consider a collection of fields satisfying the condition of Γ -locality, i.e.

$$(z-w)_S^n [a(z), b(w)] = 0 \quad (0.2)$$

The study of algebraic properties of Γ -locality leads us to a formal definition of a Γ -conformal algebra given below. Property (0.2) is equivalent to the following form of the OPE (Proposition 1.1):

$$a(z)b(w) \sim \sum_{\alpha \in S} \frac{c^\alpha(w)}{z-\alpha w}. \quad (0.3)$$

We view the field $c^\alpha(w)$ as an “ α -product” of the fields $a(z)$ and $b(z)$:

$$c^\alpha(w) = (a_{(\alpha)}b)(w). \quad (0.4)$$

Introduce operators T_α ($\alpha \in \mathbb{C}^\times$) on the space of fields by

$$T_\alpha a(z) = \alpha a(\alpha z). \quad (0.5)$$

Let R be a space over \mathbb{C} of Γ -local fields which is closed under products (0.4) and invariant under operators (0.5) for all $\alpha \in \Gamma$. Then one can show (Proposition 2.1) that the following properties hold, where $a, b, c \in R$, $\alpha, \beta \in \Gamma$:

(C0) $a_{(\alpha)}b = 0$ for all but finitely many $\alpha \in \Gamma$,

- (C1) $(T_\alpha a)_{(\beta)} b = a_{(\beta\alpha)} b,$
- (C2) $a_{(\alpha)} b = -T_\alpha(b_{(\alpha^{-1})} a),$
- (C3) $[a_{(\alpha)}, b_{(\beta)}] c = (a_{(\beta^{-1}\alpha)} b)_{(\beta)} c.$

Now we observe that one can forget that Γ is a subgroup of \mathbb{C}^\times and that R is a space of fields. What remains is a (not necessarily abelian) group Γ , its representation space R and \mathbb{C} -bilinear products $a_{(\alpha)} b$ on R for each $\alpha \in \Gamma$, such that the axioms (C0)–(C3) hold. This is the basic object of our study, called a Γ -conformal algebra.

Recall that the OPE (0.1) of local fields similarly leads to the notion of a conformal algebra [K1]. However, conformal algebra and Γ -conformal algebras have dramatically different properties.

Our first result on Γ -conformal algebras is Theorem 3.1 which gives their classification in terms of admissible pairs (\mathfrak{g}, φ) , where \mathfrak{g} is a Lie algebra and φ is a homomorphism of Γ to the group $\text{Aut } \mathfrak{g}$ of automorphisms of \mathfrak{g} such that for any $a, b \in \mathfrak{g}$ one has

$$[T_\alpha a, b] = 0 \text{ for all but finitely many } \alpha \in \Gamma. \quad (0.6)$$

Of course, if Γ is a finite subgroup of $\text{Aut } \mathfrak{g}$, then (0.6) holds. The associated Γ -conformal algebra is called a twisted current conformal algebra. A more interesting example corresponds to the admissible pair $(g\ell_\infty, \mathbb{Z})$, where $g\ell_\infty$ is the Lie algebra of all $\mathbb{Z} \times \mathbb{Z}$ -matrices (a_{ij}) over \mathbb{C} with a finite number of non-zero entries and the action of \mathbb{Z} is by translations along the diagonal: $n \cdot (a_{i,j}) = (a_{i+n,j+n})$, $n \in \mathbb{Z}$. This \mathbb{Z} -conformal algebra is denoted by $gc_1(\mathbb{Z})$ for reasons explained below.

To a Γ -conformal algebra and a homomorphism χ of Γ to \mathbb{C}^\times (or, more generally, to the group of meromorphic transformations, see § 5) one canonically associates an infinite-dimensional Lie algebra. This construction produces many known (and a lot of new) examples of infinite-dimensional Lie algebras which appear in deformation of conformal field theories. For example, the Lie algebra of q -differential operators on the circle (the sin-algebra) is obtained from the \mathbb{Z} -conformal algebra $gc_1(\mathbb{Z})$ by using $\chi : \mathbb{Z} \rightarrow \mathbb{C}^\times$ defined by $\chi(n) = q^n$. Taking the same \mathbb{Z} -conformal algebra, but the homomorphism $\chi : \mathbb{Z} \rightarrow GL_2(\mathbb{C})$ defined by $\chi(n)z = z + n$, $z \in \mathbb{C}$, instead, produces the Lie algebra of pseudodifferential operators on the circle (see § 5).

Representations of Γ -conformal algebras are classified by equivariant (\mathfrak{g}, Γ) -modules satisfying a finiteness condition (Theorem 4.1). On the other

hand, to any representation π of the group Γ we canonically associate the general Γ -conformal algebra $gc(\pi, \Gamma)$, which plays the role of the general linear group in the sense that representations of a Γ -conformal algebra R correspond to homomorphisms $R \rightarrow gc(\pi, \Gamma)$ (Theorem 4.2). The most important case is when π is a free $\mathbb{C}[\Gamma]$ -module of rank N . The corresponding general Γ -conformal algebra, denoted by $gc_N(\Gamma)$, is a generalization of $gc_1(\mathbb{Z})$ mentioned above. It is an analogue of gc_N in the theory of conformal algebras [K2].

In the present paper we consider the simplest case of simple poles and of one indeterminate z . It is straightforward to generalize this to the case of multiple poles and the case of several indeterminates. This will be discussed elsewhere.

1 S -local formal distributions

Recall that a formal distribution with coefficients in a vector space U is an expression of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \text{ where } a_n \in U.$$

They form a vector space denoted by $U[[z, z^{-1}]]$. We shall use the standard notation:

$$\text{Res}_{z=0} a(z) = a_0.$$

Similarly, a formal distribution in z and w is an expression of the form

$$a(z, w) = \sum_{m, n \in \mathbb{Z}} a_{m, n} z^{-m-1} w^{-n-1}, \text{ where } a_{m, n} \in U.$$

It may be viewed as a formal distribution in z with coefficients in $U[[w, w^{-1}]]$.

Recall that the formal δ -function is the following formal distribution in z and w with coefficients in \mathbb{C} :

$$\delta(z - w) = \sum_{n \in \mathbb{Z}} z^{n-1} w^{-n}.$$

It has the following two basic properties:

$$(z - w)\delta(z - w) = 0 \tag{1.1}$$

$$\text{Res}_{z=0} a(z)\delta(z - w) = a(w) \text{ for } a(z) \in U[[z, z^{-1}]]. \tag{1.2}$$

We shall use the formal distribution

$$\delta(\alpha z - \beta w) = \sum_{n \in \mathbb{Z}} (\alpha z)^{n-1} (\beta w)^{-n}, \quad \alpha, \beta \in \mathbb{C}^\times,$$

and the following obvious property

$$\delta(\alpha z - \beta w) = \alpha^{-1} \delta(z - \alpha^{-1} \beta w) = \beta^{-1} \delta(\alpha \beta^{-1} z - w). \quad (1.3)$$

We shall use also the following equality of distribution in z_1, z_2 and z_3 :

$$\delta(z_1 - z_2) \delta(z_2 - z_3) = \delta(z_1 - z_2) \delta(z_1 - z_3) \quad (1.4)$$

Let S be an N -element set of (distinct) non-zero complex numbers. We shall use the following notation:

$$(z - w)_S^N = \prod_{\alpha \in S} (z - \alpha w) \quad (1.5)$$

Definition 1.1 *Two formal distributions $a(z)$ and $b(z)$ with coefficients in Lie algebra \mathfrak{g} are called S -local if in $\mathfrak{g}[[z, z^{-1}, w, w^{-1}]]$ one has:*

$$(z - w)_S^N [a(z), b(w)] = 0. \quad (1.6)$$

Note that S -local formal distributions are S' -local for any finite subset S' of \mathbb{C}^\times containing S .

Proposition 1.1 *If $a(z)$ and $b(z)$ are S -local formal distributions, then there exists a unique decomposition of the form*

$$[a(z), b(w)] = \sum_{\alpha \in S} c_\alpha(w) \delta(z - \alpha w). \quad (1.7)$$

The formal distributions $c_\alpha(w)$ are given by the formula

$$c_\alpha(w) = \operatorname{Res}_{z=0} P_{S,\alpha} \left(\frac{z}{w} \right) [a(z), b(w)], \quad (1.8)$$

where

$$P_{S,\alpha}(u) = \prod_{\substack{\beta \in S \\ \beta \neq \alpha}} \frac{u - \beta}{\alpha - \beta} \quad (1.9)$$

Proof of this proposition is immediate by the following lemma.

Lemma 1.1 (a) *Each formal distribution $a(z, w)$ can be uniquely written in the form:*

$$a(z, w) = \sum_{\alpha \in S} c_\alpha(w) \delta(z - \alpha w) + b(z, w) \quad (1.10)$$

where

$$c_\alpha(w) = \operatorname{Res}_{z=0} P_{S,\alpha} \left(\frac{z}{w} \right) a(z, w) \quad (1.11)$$

and

$$\operatorname{Res}_{z=0} P_{S,\alpha} \left(\frac{z}{w} \right) b(z, w) = 0 \quad \text{for all } \alpha \in S. \quad (1.12)$$

(b) *If $b(z, w)$ satisfies (1.12) and $(z - w)_S^N b(z, w) = 0$, then $b(z, w) \equiv 0$.*

Proof: Consider the following operator on the space of formal distribution in z and w :

$$\pi_S a(z, w) = \sum_{\alpha \in S} c_\alpha(w) \delta(z - \alpha w),$$

where $c_\alpha(w)$ are given by (1.11). It is clear using (1.2) that $\pi_S^2 = \pi_S$, which implies (a). Furthermore, suppose that $(z - w)_S^N b(z, w) = 0$. It follows from [K1, corollary 2.2] that for each $\alpha \in S$:

$$P_{S,\alpha} b(z, w) = d_\alpha(w) \delta(z - \alpha w)$$

for some formal distribution $d_\alpha(w)$. If also (1.12) holds, then $\operatorname{Res}_{z=0} d_\alpha(w) \delta(z - \alpha w) = 0$, hence $d_\alpha(w) = 0$ by (1.2), and

$$P_{S,\alpha} \left(\frac{z}{w} \right) b(z, w) = 0 \quad \text{for all } \alpha \in S. \quad (1.13)$$

This implies that $b(z, w) = 0$ since the polynomials, $P_{S,\alpha}$ are relatively prime. (Indeed $\sum_{\alpha \in S} P_{S,\alpha} u_\alpha = 1$ for some polynomials u_α . Multiplying both sides of (1.13) by $u_\alpha \left(\frac{z}{w} \right)$ and summing over α , we get $b(z, w) = 0$.) This proves (b). \square

Corollary 1.1 *The formal distributions $a(z)$ and $b(z)$ are S -local iff there exists a decomposition of the form (1.7).*

Proof: Follows from Proposition 1.1 and (1.1). \square

Remark 1.1 The coefficients $c_\alpha(w)$ are independent of the choice of S for which (1.6) holds. This follows from the uniqueness of the decomposition (1.10).

Remark 1.2 Formula (1.7), written out in modes, looks as follows:

$$[a_m, b_n] = \sum_{\alpha \in S} \alpha^m c_{\alpha, m+n}.$$

2 α -products of Γ -local formal distributions

Let Γ be a subgroup of \mathbb{C}^\times . Two formal distributions $a(z)$ and $b(z)$ are called Γ -local if they are S -local for some finite subset S of Γ . For $\alpha \in \Gamma$ define the α -product $(a_{(\alpha)}b)(w)$ as the coefficient $c_\alpha(z)$ in the expansion (1.7). In other words,

$$(a_{(\alpha)}b)(w) = \text{Res}_{z=0} P_{S,\alpha} \left(\frac{z}{w} \right) [a(z), b(w)]. \quad (2.1)$$

Due to Remark 1.1, the α -product is independent of the choice of S .

For $\beta \in \Gamma$ introduce the following operator:

$$T_\beta(a(z)) = \beta a(\beta z).$$

It is clear that T_β preserves Γ -locality.

We collect below the main properties of α -products.

Proposition 2.1 *For Γ -local formal distributions one has the following properties ($\alpha, \beta \in \Gamma$):*

- (a) *(Translation invariance) $(T_\alpha a)_{(\beta)} b = a_{(\alpha\beta)} b$, and $a_{(\beta)} T_\alpha b = T_\alpha (a_{(\beta\alpha^{-1})} b)$.*
- (b) *(Skewsymmetry) $a_{(\alpha)} b = -T_\alpha (b_{(\alpha^{-1})} a)$.*
- (c) *(Jacobi identity) $a_{(\alpha)} (b_{(\beta)} c) = a_{(\alpha\beta^{-1})} b_{(\beta)} c + b_{(\beta)} (a_{(\alpha)} c)$ provided that all the suitable localities hold.*

Proof:

(a) We have:

$$\begin{aligned}
[T_\alpha a(z), b(w)] &= \sum_{\beta \in \Gamma} \left((T_\alpha a)_{(\beta)} b \right) (w) \delta(z - \beta w), \\
[T_\alpha a(z), b(w)] &= [\alpha a(\alpha z), b(w)] = \sum_{\gamma \in \Gamma} \alpha(a_{(\gamma)} b)(w) \delta(\alpha z - \gamma w) \\
&= \sum_{\gamma \in \Gamma} (a_{(\gamma)} b)(w) \delta(z - \alpha^{-1} \gamma w).
\end{aligned}$$

Comparing coefficients, we get the first equation of (a).

(b) We have using (1.7) and (1.3):

$$\begin{aligned}
[a(z), b(w)] &= -[b(w), a(z)] = - \sum_{\beta \in \Gamma} (b_{(\beta)} a)(z) \delta(w - \beta z) \\
&= - \sum_{\beta \in \Gamma} \beta (b_{(\beta^{-1})} a)(\beta w) \delta(z - \beta w)
\end{aligned}$$

Comparing with (1.7), we obtain (b). The second equation of (a) follows from the first and (b)

(c) We have:

$$\begin{aligned}
[a(z), [b(w), c(t)]] &= \sum_{\alpha, \beta \in \Gamma} \left(a_{(\alpha)} (b_{(\beta)} c) \right) (t) \delta(z - \alpha t) \delta(w - \beta t), \\
[[a(z), b(w)], c(t)] &= \sum_{\alpha, \beta \in \Gamma} \left((a_{(\alpha\beta^{-1})} b)_{(\beta)} c \right) (t) \delta(z - \alpha\beta^{-1} w) \delta(w - \beta t) \\
&= \sum_{\alpha \in \Gamma} \left((a_{(\alpha\beta^{-1})} b)_{(\beta)} c \right) (t) \delta(z - \alpha t) \delta(w - \beta t).
\end{aligned}$$

We have used here (1.3) and (1.4). We also have:

$$[b(w), [a(z), c(t)]] = \sum_{\alpha, \beta \in \Gamma} b_{(\beta)} (a_{(\alpha)} c)(t) \delta(w - \beta t) \delta(z - \alpha t).$$

Equating the first equality with the sum of the remaining two and using Jacobi identity in \mathfrak{g} , we obtain equality (c) due to linear independence of $\delta(w - \beta t) \delta(z - \alpha t)$ for $\alpha, \beta \in \Gamma$. \square

3 Γ -conformal algebras and Lie algebras of Γ -local formal distributions

The above considerations motivate the following definitions. Let Γ be a group and let $\mathbb{C}[\Gamma]$ be the group algebra of Γ with basis T_α ($\alpha \in \Gamma$), so that $T_\alpha T_\beta = T_{\alpha\beta}$, $T_1 = 1$. Since Γ is not necessarily abelian we must distinguish the case of left and right $\mathbb{C}[\Gamma]$ -modules.

Definition 3.1 *A left $\mathbb{C}[\Gamma]$ -module R is called a left Γ -conformal algebra if it is equipped with a \mathbb{C} -bilinear product $a_{(\alpha)}b$ for each $\alpha \in \Gamma$ such that the following axioms hold ($a, b, c \in R$, $\alpha, \beta \in \Gamma$):*

- (C0) $a_{(\alpha)}b = 0$ for all but finitely many $a \in \Gamma$,
- (C1) $(T_\alpha a)_{(\beta)}b = a_{(\beta\alpha)}b$,
- (C2) $a_{(\alpha)}b = -T_\alpha(b_{(\alpha^{-1})}a)$,
- (C3) $a_{(\alpha)}(b_{(\beta)}c) = (a_{(\beta^{-1}\alpha)}b)_{(\beta)}c + b_{(\beta)}(a_{(\alpha)}c)$.

A right $\mathbb{C}[\Gamma]$ -module R is called a *right Γ -conformal algebra* if axioms (C1), (C2) and (C3) are replaced by:

- (C1)_R $(aT_\alpha)_{(\beta)}b = a_{(\alpha\beta)}b$,
- (C2)_R $a_{(\alpha)}b = -(b_{(\alpha^{-1})}a)T_\alpha$,
- (C3)_R $a_{(\alpha)}(b_{(\beta)}c) = (a_{(\alpha\beta^{-1})}b)_{(\beta)}c + b_{(\beta)}(a_{(\alpha)}c)$.

Unless otherwise specified, we will consider left Γ -conformal algebras and will drop the adjective left.

Remark 3.1 (a) Properties (C1) and (C2) (resp. (C1)_R and (C2)_R) imply

$$(C1') \quad a_{(\beta)}T_\alpha b = T_\alpha(a_{(\alpha^{-1}\beta)}b) \quad (\text{resp. } (C1')_R \quad a_{(\beta)}(bT_\alpha) = (a_{(\beta\alpha^{-1})}b)T_\alpha)$$

(b) (C1) and (C1') (resp. (C1)_R and (C1')_R) imply

$$(T_\alpha a)_{(\beta)}(T_\alpha b) = T_\alpha(a_{(\alpha^{-1}\beta\alpha)}b) \quad (\text{resp. } (aT_\alpha)_{(\beta)}(bT_\alpha) = (a_{(\alpha\beta\alpha^{-1})}b)T_\alpha).$$

In particular, each T_α is an automorphism of the product $a_{(1)}b$ for each $\alpha \in \Gamma$.

- (c) With respect to the product $a_{(1)}b$, R is a Lie algebra over \mathbb{C} . Due to (C1) all other products are expressed via the 1-product:

$$a_{(\alpha)}b = (T_\alpha a)_{(1)}b$$

Due to Remark 3.1, we can associate to a Γ -conformal algebra R a pair (\mathfrak{g}, φ) , where \mathfrak{g} is a Lie algebra over \mathbb{C} whose underlying space is R and bracket is $[a, b] = a_{(1)}b$, and φ is a homomorphism of Γ to the group $\text{Aut}(\mathfrak{g})$ such that

$$[T_\alpha a, b] = 0 \quad \text{for all but finitely many } \alpha \in \Gamma. \quad (3.1)$$

We call such (\mathfrak{g}, φ) an *admissible pair*. Conversely, given an admissible pair (\mathfrak{g}, φ) we denote by $R(\mathfrak{g}, \varphi)$ the $\mathbb{C}[\Gamma]$ -module \mathfrak{g} with \mathbb{C} -bilinear products

$$a_{(\alpha)}b = [T_\alpha a, b] \quad (3.2)$$

It is straightforward to check that $R(\mathfrak{g}, \varphi)$ is a Γ -conformal algebra.

We have proved the following result:

Theorem 3.1 *Γ -conformal algebras are classified by admissible pairs (\mathfrak{g}, φ) .*

Fix now a homomorphism $\chi : \Gamma \rightarrow \mathbb{C}^\times$. Let \mathfrak{s} be a Lie algebra and let R be a family of pairwise $\chi(\Gamma)$ -local formal distributions with coefficients in \mathfrak{s} , which span \mathfrak{s} over \mathbb{C} . Assume that R is stable under all T_α and all α -products for $a \in \chi(\Gamma)$. Then \mathfrak{s} is called a *Lie algebra of $\chi(\Gamma)$ -local formal distributions*.

It follows from Proposition 2.1 that R is a $\chi(\Gamma)$ -conformal algebra. Conversely, given a Γ -conformal algebra R , we associate to it a Lie algebra $\mathfrak{s}(R)$ of $\chi(\Gamma)$ -local formal distributions as follows. Consider a vector space over \mathbb{C} with the basis a_n , where $a \in R$ and $n \in \mathbb{Z}$, and denote by $\mathfrak{s}(R, \chi)$ the quotient of this space by the \mathbb{C} -span of elements of the form ($n \in \mathbb{Z}$):

$$(\lambda a + \mu b)_n - \lambda a_n - \mu b_n, \quad \text{where } \lambda, \mu \in \mathbb{C}, \quad a, b \in R, \quad (3.3a)$$

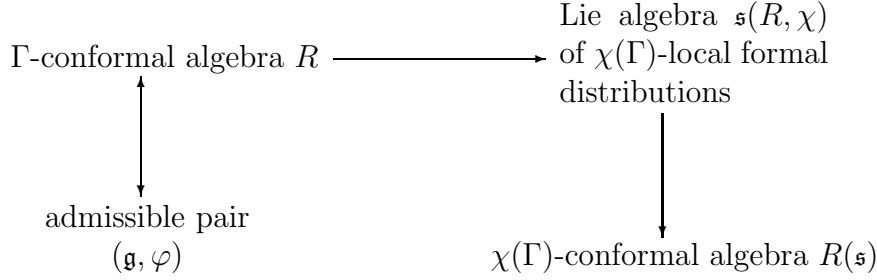
$$(T_\alpha a)_n - \chi(\alpha)^{-n} a_n, \quad \text{where } \alpha \in \Gamma, \quad a \in R. \quad (3.3b)$$

Then it is straightforward to check that the formula (cf. Remark 1.2)

$$[a_m, b_n] = \sum_{\gamma \in \Gamma} \chi(\gamma)^m (a_{(\gamma)} b)_{m+n} \quad (3.4)$$

gives a well-defined structure of a Lie algebra on $\mathfrak{s}(R, \chi)$ of $\chi(\Gamma)$ -local formal distributions $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, $a \in R$. The associated $\chi(\Gamma)$ -conformal algebra is denoted by $R(\mathfrak{s})$.

The relations between the constructed objects can be summarized by the following diagram:



Now we turn to the most important examples of Γ -conformal algebras and related constructions.

Example 3.1 Let $(\mathfrak{g}, \mathbb{Z}/N\mathbb{Z})$ be a finite-dimensional (simple) Lie algebra \mathfrak{g} with the action of the group $\Gamma = \mathbb{Z}/N\mathbb{Z}$ by automorphisms of \mathfrak{g} . Due to this action T_α , $\alpha \in \Gamma$, we have the eigenspace decomposition ($\hat{\Gamma}$ is the group of characters of Γ):

$$\mathfrak{g} = \bigoplus_{j \in \hat{\Gamma}} \mathfrak{g}_j.$$

The Γ -conformal algebra $R(\mathfrak{g}, \varphi)$ is defined as $\mathbb{C}[\Gamma]$ -module with underlying space \mathfrak{g} and \mathbb{C} -bilinear products $a_{(\alpha)}b = [T_\alpha a, b]$, $a, b \in \mathfrak{g}$, $\alpha \in \mathbb{Z}/N\mathbb{Z}$. Fix a homomorphism $\chi : \Gamma \rightarrow \mathbb{C}^\times$ such that $\chi(n) = \epsilon^n$, where $\epsilon \in \mathbb{C}$ is an N -th root of 1. The corresponding Lie algebra of $\chi(\Gamma)$ -local formal distributions $\mathfrak{s}(R, \chi)$ is nothing but the twisted affine algebra $[K]$ with the commutation relations:

$$[a_m, b_n] = \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \epsilon^{km} [T_k a, b]_{m+n}$$

Equivalently:

$$[a(z), b(w)] = \sum_{\alpha \in \mathbb{Z}/N\mathbb{Z}} [T_\alpha a, b](w) \delta(z - \alpha w).$$

Suppose that $a \in \mathfrak{g}_j$, then $T_k a = \epsilon^{kj} a$ and

$$[a_m, b_n] = \begin{cases} 0 & \text{if } m \neq -j \bmod N \\ N[a, b]_{m+n} & \text{if } m = -j \bmod N \end{cases}$$

Thus we put $a_m = 0$ for $m \neq -j \bmod N$, which is the same as to take a quotient by relations (3.3b).

Example 3.2 Consider $\Gamma = \mathbb{Z}$ and the admissible pair (\mathfrak{g}, φ) , where $\mathfrak{g} = g\ell_\infty$ is the Lie algebra of all $\mathbb{Z} \times \mathbb{Z}$ matrices over \mathbb{C} with finitely many non-zero entries. Let E_{ij} ($i, j \in \mathbb{Z}$) be its standard basis, with the usual relations:

$$[E_{ij}, E_{kl}] = \delta_{j,k} E_{il} - \delta_{\ell,i} E_{kj}.$$

Let T be the image of $1 \in \mathbb{Z}$ under φ . Consider two different actions of T on $g\ell_\infty$.

a) $T : E_{ij} \rightarrow E_{i+1, j+1}$.

In this case $g\ell_\infty$ is a free $\mathbb{C}[T, T^{-1}]$ -module with the generators $A^m = E_{0,m}$, $m \in \mathbb{Z}$. As follows from Definition 3.1 of a Γ -conformal algebra, it is sufficient to define the Γ -products only on generators of the $\mathbb{C}[\Gamma]$ -module. So, for the case under consideration we have:

$$\begin{aligned} A_{(r)}^m A^n &= [T^r A^m, A^n] = [E_{r, m+r}, E_{0,n}] \\ &= \delta_{r, -m} T^{-m} A^{m+n} - \delta_{r,n} A^{m+n}, \quad r \in \mathbb{Z} \end{aligned} \quad (3.5)$$

Take a homomorphism $\chi = \chi_q : \mathbb{Z} \rightarrow \mathbb{C}^\times$ defined by $\chi_q(1) = q \in \mathbb{C}^\times$. The corresponding Lie algebra $\mathfrak{s}(R, \chi)$ is the Lie algebra with the basis A_k^m , $m, k \in \mathbb{Z}$, and the commutation relations:

$$[A_k^m, A_\ell^n] = \sum_{r \in \mathbb{Z}} q^{rk} (A_{(r)}^m A^n)_{k+\ell} = (q^{m\ell} - q^{kn}) A_{k+\ell}^{m+n}. \quad (3.6)$$

This is the Lie algebra of q -pseudodifferential operators on the circle (sin-algebra).

b) Now define the action of $\varphi(1) = \tilde{T}$ by $\tilde{T}(E_{i,j}) = -E_{j+1, i+1}$. We have $\tilde{T} = \epsilon T$, where ϵ is an order 2 automorphism of $g\ell_\infty$ defined by $\epsilon(E_{ij}) = -E_{ji}$. In this case $g\ell_\infty$ is a free $\mathbb{C}[\tilde{T}, \tilde{T}^{-1}]$ -module with generators $A^m = E_{0,m}$ and $B^m = \epsilon(A^m) = -E_{m,0}$, $m \in \mathbb{Z}_+$. Let us compute the r -product, $r \in \mathbb{Z}$, for the elements of the basis of $\mathbb{C}[\Gamma]$ -module. Suppose $n - m \geq 0$ (for $n - m < 0$ we can use axiom (C2)):

$$\begin{aligned} A_{(r)}^m A^n &= [\tilde{T}^r A^m, A^n] \\ &= \begin{cases} \delta_{r, -m} T^{-m} A^{m+n} - \delta_{r,m} A^{m+n}, & m, n \text{ even;} \\ \delta_{r, -m} T^{-m} A^{m+n} + \delta_{r, n-m} A^{n-m}, & m \text{ even, } n \text{ odd;} \\ -\delta_{r,n} A^{m+n} + \delta_{r, n-m} A^{n-m}, & m \text{ odd, } n \text{ even} \\ 0 & m, n \text{ odd.} \end{cases} \end{aligned}$$

$$\begin{aligned}
B_{(r)}^m B^n &= \epsilon[\tilde{T}^r A^m, A^n] \\
B_{(r)}^m A^n &= \begin{cases} -\delta_{r,0} \tilde{T}^m A^{n-m} + \delta_{r,n-m} A^{n-m}, & m, n \text{ even}; \\ -\delta_{r,0} \tilde{T}^m A^{n-m} - \delta_{r,n} A^{m+n}, & m \text{ even}, n \text{ odd}; \\ -\delta_{r,0} \tilde{T}^m B^{n-m} + \delta_{r,-m} \tilde{T}^{-m} B^{n+m}, & m \text{ odd}, n \text{ even}; \\ -\delta_{r,0} \tilde{T}^n B^{n-m} + \delta_{r,n-m} A^{n+m} \\ \quad + \delta_{r,-m} \tilde{T}^{-m} B^{n+m} - \delta_{r,n} A^{m+n}, & m, n \text{ odd}. \end{cases}
\end{aligned}$$

The corresponding Lie algebra $\mathfrak{s}(R, \chi_q)$ is a Lie algebra with basis $\{A_k^m\}$ and $\{B_\ell^n\}$, $m, n \in \mathbb{Z}_+$, $k, \ell \in \mathbb{Z}$. From (3.4) and (3.5) we have for $n - m \geq 0$:

$$\begin{aligned}
[A_k^m, A_\ell^n] &= \begin{cases} (q^{m\ell} - q^{nk}) A_{k+\ell}^{m+n}, & m, n \text{ even}, \\ q^{m\ell} A_{k+\ell}^{m+n} + q^{k(n-m)} A_{k+\ell}^{n-m}, & m \text{ even}, n \text{ odd}; \\ -q^{nk} A_{k+\ell}^{m+n} + q^{k(n-m)} A_{k+\ell}^{n-m}, & m \text{ odd}, n \text{ even}; \\ 0 & m, n \text{ odd}, \end{cases} \\
[B_k^m, B_\ell^n] &= \text{the same as for } [A_n^m, A_\ell^n] \text{ with change } A \rightarrow B. \\
[B_k^m, A_\ell^n] &= \begin{cases} q^{-mk} (q^{kn} - q^{-m\ell}) A_{k+\ell}^{n-m}, & m, n \text{ even}, \\ -q^{-m(k+\ell)} A_{k+\ell}^{n-m} - q^{kn} A_{k+\ell}^{m+n}, & m \text{ even}, n \text{ odd}; \\ -q^{-m(k+\ell)} B_{k+\ell}^{n-m} + q^{m\ell} B_{k+\ell}^{n+m}, & m \text{ odd}, n \text{ even}; \\ -q^{-m(k+\ell)} B_{k+\ell}^{n-m} + q^{k(n-m)} A_{k+\ell}^{n-m} \\ \quad + q^{m\ell} B_{k+\ell}^{n+m} - q^{nk} A_{k+\ell}^{n+m}, & m, n \text{ odd}. \end{cases}
\end{aligned}$$

This Lie algebra is apparently new. It contains as a subalgebra the sin-algebra $\{\tilde{A}_n^k\}$, where $\tilde{A}_n^k = A_k^n$, $n > 0$, n even; $A_k^n = q^{nk} B_k^{-n}$, $n < 0$, n even; and the subalgebra isomorphic to B -series of Lie algebras of quantum torus (see [G-K-L]) for n odd.

Example 3.3 Consider the group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}$ and admissible pair $(g\ell_\infty, \varphi)$, where homomorphism $\varphi : \Gamma \rightarrow \text{Aut } g\ell_\infty$ is determined by two elements ϵ and T as in Example 3.2 (b). With respect to the Γ -action $g\ell_\infty$ is a free module with the generators $B^m = E_{0,m}$, $m \geq 0$. The corresponding conformal algebra $R(g\ell_\infty, \varphi)$ is defined by the Γ -products:

$$\begin{aligned}
B_{(r)}^m B^n &= [T^r B^m, B^n] = \delta_{r,-m} T^{-m} B^{n+m} - \delta_{n,r} B^{m+r}, \\
B_{(\epsilon r)}^m B^n &= [\epsilon T^r B^m, B^n] = \begin{cases} -\delta_{r,0} T^m B^{n-m} + \delta_{r,n-m} B^{n-m}, & n - m \geq 0 \\ \delta_{r,0} \epsilon T^n B^{n-m} - \delta_{r,n-m} B^{m-n}, & n - m < 0. \end{cases}
\end{aligned} \tag{3.7}$$

Fix the homomorphism $\chi : \Gamma \rightarrow \mathbb{C}^\times$ such that $\chi(\epsilon) = -1$, $\chi(1) = q \in \mathbb{C}^\times$. The Lie algebra $\mathfrak{s}(R(g\ell_\infty, \varphi), \chi)$ is a Lie algebra with the basis B_k^m , $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, and commutation relations:

$$\begin{aligned} [B_k^m, B_\ell^n] &= (q^{m\ell} - q^{nk})B_{k+\ell}^{m+n} \\ &\quad - \begin{cases} (-1)^k(q^{k(n-m)} - q^{-m(k+\ell)})B_{k+\ell}^{n-m}, & n - m > 0, \\ (-1)^k((-q)^{-n(k+\ell)} \\ \quad - (-1)^{-n(k+\ell)}(-q)^{m(k+\ell)})B_{k+\ell}^{m-n}, & n - m < 0 \end{cases} \end{aligned}$$

After the renormalization of the basis $\tilde{B}_k^m = q^{\frac{mk}{2}} B_k^m$ we get exactly the commutation relations for B -series of sin Lie algebras, introduced in [G-K-L]:

$$[\tilde{B}_k^m, \tilde{B}_\ell^n] = \left(q^{\frac{m\ell - nk}{2}} - q^{\frac{-(m\ell - nk)}{2}} \right) \tilde{B}_{k+\ell}^{n+m} - (-1)^k \left(q^{\frac{kn + m\ell}{2}} - q^{-\frac{m\ell + kn}{2}} \right) \tilde{B}_{k+\ell}^{n-m}$$

4 Representations of Γ -conformal algebras and the general Γ -conformal algebra $gc(\pi, \Gamma)$.

Definition 4.1 A (left) module over a Γ -conformal algebra R is a $\mathbb{C}[\Gamma]$ -module M with a \mathbb{C} -linear map $a \mapsto a_\alpha^M$ of R to $\text{End}_{\mathbb{C}} M$ for each $\alpha \in \Gamma$ such that the following properties hold (here $a, b \in R$, $\alpha, \beta \in \Gamma$ and the action of Γ on M is denoted by $\alpha \mapsto T_\alpha^M$):

$$(M0) \quad a_{(\alpha)}^M v = 0 \text{ for } v \in M \text{ and all but finitely many } \alpha.$$

$$(M1) \quad (T_\alpha a)_{(\beta)}^M = a_{\beta\alpha}^M, \quad a_{(\beta)}^M T_\alpha^M = T_\alpha^M a_{(\alpha^{-1}\beta)}^M,$$

$$(M2) \quad [a_{(\alpha)}^M, b_{(\beta)}^M] = (a_{(\beta^{-1}\alpha)} b)_{(\beta)}^M.$$

For example, $a \mapsto a_{(\alpha)}^R$ is the *adjoint representation* of R on itself.

It follows from (M1) that

$$(T_\alpha a)_{(\beta)}^M (T_\alpha^M v) = T_\alpha^M (a_{(\alpha^{-1}\beta\alpha)}^M v), \quad v \in M. \quad (4.1)$$

This leads us to

Theorem 4.1 *Let R be a Γ -conformal algebra and let (\mathfrak{g}, φ) be the associated admissible pair. Then R -modules M are classified by equivariant (\mathfrak{g}, Γ) -modules such that for any $a \in \mathfrak{g}$ and $v \in M$ one has:*

$$(T_\alpha^{\mathfrak{g}} a)v = 0 \text{ for all but finitely many } \alpha \in \Gamma. \quad (4.2)$$

Proof: It follows from (M2) that $a \mapsto a_{(1)}$ is a representation of \mathfrak{g} . It satisfies (4.1) due to (M2), and it is equivariant, i.e. $(T_\alpha^{\mathfrak{g}} a)(T_\alpha^M v) = T_\alpha^M(av)$ due to (4.1). Conversely, given an equivariant (\mathfrak{g}, Γ) -module M , it becomes an R -module by letting

$$a_\alpha^M = (T_\alpha^{\mathfrak{g}} a)_{(1)}^M \text{ on } M.$$

□

Consider a representation $\alpha \mapsto T_\alpha$ of the group Γ in a vector space V over \mathbb{C} . We define a Γ -conformal endomorphism of V as a collection $a = \{a_{(\beta)}\}_{\beta \in \Gamma}$ of \mathbb{C} -endomorphisms of V such that

$$a_{(\beta)} T_\alpha = T_\alpha a_{(\alpha^{-1}\beta)}, \quad \alpha, \beta \in \Gamma, \quad (4.3)$$

and for each $v \in V$

$$a_{(\alpha)} v = 0 \text{ for all but finitely many } \alpha \in \Gamma. \quad (4.4)$$

We denote the set of all Γ -conformal endomorphisms of V by $gc(V, \Gamma)$.

Define a $\mathbb{C}[\Gamma]$ -module structure on the space $gc(V, \Gamma)$ by

$$(T_\alpha a)_{(\beta)} = a_{(\beta\alpha)}, \quad \alpha, \beta \in \Gamma, \quad (4.5)$$

and α -product for each $\alpha \in \Gamma$ by

$$(a_{(\alpha)} b)_{(\beta)} = [a_{(\beta\alpha)}, b_{(\beta)}]. \quad (4.6)$$

It is immediate to check that axioms (C1), (C2) and (C3) of a Γ -conformal algebra hold (though axiom (C0) probably doesn't hold in general).

It is clear that we have by definition

Proposition 4.1 *To give a Γ -module V a structure of a module over a Γ -conformal algebra R is the same as to give a homomorphism $R \rightarrow gc(V, \Gamma)$ (i.e. a $\mathbb{C}[\Gamma]$ -module homomorphism preserving all α -products).*

We will show now that axiom (C0) holds, hence $gc(V, \Gamma)$ is a Γ -conformal algebra, provided that the $\mathbb{C}[\Gamma]$ -module V is finitely generated. Towards this end we shall give a different construction of $gc(V, \Gamma)$ in the case when V is a free $\mathbb{C}[\Gamma]$ -module of rank 1.

Let $gc_1(\Gamma) = \bigoplus_{r \in \Gamma} \mathbb{C}[\Gamma]a^r$ be a free $\mathbb{C}[\Gamma]$ -module with free generators a^r labeled by elements of Γ . For each $\alpha \in \Gamma$ define the α -product by

$$a^r_{(\alpha)}a^s = \delta_{\alpha, r-1}T_{r-1}a^{rs} - \delta_{\alpha, s}a^{sr}, \quad (4.7)$$

extending to the whole R by axioms (C1) and (C1').

Theorem 4.2 (a) *The $\mathbb{C}[\Gamma]$ -module $gc_1(\Gamma)$ with products (4.7) is a Γ -conformal algebra.*

(b) *Let $V = \mathbb{C}[\Gamma]v$ be a free $\mathbb{C}[\Gamma]$ -module of rank 1. Define the action of $gc_1(\Gamma)$ on V by letting*

$$(a^s)_{(\alpha)}^V v = \delta_{\alpha, s-1}T_{\alpha}v \quad (4.8)$$

and extending to V by (M1). This gives V a structure of a $gc_1(\Gamma)$ -module.

(c) *The $gc_1(\Gamma)$ -module $V = \mathbb{C}[\Gamma]v$ contains all Γ -conformal endomorphisms of V . In particular, the Γ -conformal algebras $gc(V, \Gamma)$ and $gc_1(\Gamma)$ are isomorphic.*

Proof: To prove (a) we need to check that the axioms (C0)–(C3) are satisfied. (C0) is obvious. (C1) and (C2) come from the definition (4.7) of $gc_1(\Gamma)$. Check the skew symmetry (C2):

$$\begin{aligned} (a^r_{(\alpha)}a^s)_{(\beta)} &= \delta_{\alpha, r-1}(T_{r-1}a^{rs})_{(\beta)} - \delta_{\alpha, s}a^{sr}_{(\beta)}, \\ T_{\alpha}(a^s_{(\alpha-1)}a^r)_{\beta} &= \delta_{\alpha, s}T_{\alpha s-1}a^{sr} - \delta_{\alpha, r-1}T_{r-1}a^{rs} = -(\delta_{\alpha, r-1}T_{r-1}a^{rs} - \delta_{\alpha, s}a^{sr}). \end{aligned}$$

For (C3) we need to check that both sides of

$$a^m_{(\beta\alpha)}(a^r_{(\beta)}a^s) = (a^m_{(\alpha)}a^r)_{(\beta)}a^s + a^r_{(\beta)}(a^m_{(\beta\alpha)}a^s)$$

are equal.

$$\begin{aligned} \text{LHS: } a^m_{(\beta\alpha)}(a^r_{(\beta)}a^s) &= a^m_{(\beta\alpha)}(\delta_{\beta, r-1}T_{r-1}T_{r-1} - \delta_{\beta, s}a^{sr}) \\ &= \delta_{\beta, r-1}T_{r-1}a^m_{(\alpha)}a^{rs} - \delta_{\beta, s}a^m_{(\beta\alpha)}a^{sr} \\ &= \delta_{\beta, r-1}\delta_{\alpha, m-1}T_{(mr)-1}a^{mrs} - \delta_{\beta, r-1}\delta_{\alpha, rs}T_{r-1}a^{rsm} \\ &\quad - \delta_{\beta, s}\delta_{\beta\alpha, m-1}T_{m-1}a^{msr} + \delta_{\beta, s}\delta_{\beta\alpha, sr}a^{srm} \end{aligned}$$

$$\begin{aligned}
\text{RHS: } (a_{(\alpha)}^m a^r)_{(\beta)} a^s + a_{(\beta)}^r (a_{(\beta\alpha)}^m a^s) \\
= \delta_{\alpha, m-1} \delta_{\beta\alpha, (mr)^{-1}} T_{(mr)^{-1}} a^{mrs} - \delta_{\alpha_1 m-1} \delta_{\beta\alpha, s} a^{smr} \\
- \delta_{\alpha, r} \delta_{\beta, (rm)^{-1}} a^{rms} + \delta_{\alpha, r} \delta_{\beta, s} a^{srm} \\
+ \delta_{\beta\alpha, m-1} \delta_{\beta, r} T_{(rm)^{-1}} a^{rms} - \delta_{\beta\alpha, m-1} \delta_{\alpha^{-1}, ms} T_{m^{-1}} a^{msr} \\
- \delta_{\beta\alpha, s} \delta_{\beta, r-1} a^{rsm} + \delta_{\beta\alpha, s} \delta_{\beta, sm} a^{smr},
\end{aligned}$$

and we have LHS = RHS.

To prove (b) we need to check only axiom (M2) on V :

$$(a_{(\alpha)}^r a^s)_{(\beta)} v = [a_{(\beta\alpha)}^r, a_{(\beta)}^s] v.$$

$$\begin{aligned}
\text{LHS: } (a_{(\alpha)}^r a^s)_{(\beta)} v &= (\delta_{\alpha, r-1} a_{\beta\alpha}^{rs} - \delta_{\alpha, s} a_{(\beta)}^{sr}) v \\
&= (\delta_{\alpha, r-1} \delta_{\beta, s-1} T_{(rs)^{-1}} - \delta_{\alpha, s} \delta_{\beta(s2)^{-1}} T_{\beta}) v
\end{aligned}$$

$$\begin{aligned}
\text{RHS: } [a_{(\beta\alpha)}^r, a_{(\beta)}^s] v &= (\delta_{\beta, s-1} T_{s^{-1}} a_{(\alpha)}^r - \delta_{\beta\alpha, r-1} T_{r^{-1}} a_{(\alpha^{-1})}^s) v \\
&= (\delta_{\beta, s-1} \delta_{\alpha, r-1} T_{(rs)^{-1}} - \delta_{\beta\alpha, r-1} \delta_{\alpha, s} T_{(sr)^{-1}}) v
\end{aligned}$$

so LHS = RHS. Finally, (c) is clear. \square

Corollary 4.1 *If V is a finitely generated $\mathbb{C}[\Gamma]$ -module, then $gc(V, \Gamma)$ is a conformal algebra.*

Proof: We may assume that $V = \mathbb{C}[\Gamma]^{\oplus N}$ is a free $\mathbb{C}[\Gamma]$ -module of rank N . Then $gc(V, \Gamma)$ may be viewed as a $\mathbb{C}[\Gamma]$ -submodule of $gc_1(\Gamma^N)$ using the diagonal homomorphism $\mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma]^{\oplus N}$. Corollary now follows from Theorem 4.2a. \square

If V is a free $\mathbb{C}[\Gamma]$ -module of rank N , we use notation $gc_N(\Gamma) = gc(V, \Gamma)$.

Example 4.1 Consider $\Gamma = \mathbb{Z}^N$. For the $gc_1(\mathbb{Z}^N)$ defined by (4.7) the corresponding Lie algebra (\mathfrak{g}, φ) is a Lie algebra with generators $a_\alpha^\beta = T_\alpha a^\beta$, $\alpha, \beta \in \mathbb{Z}^N$ and commutation relations:

$$[a_\alpha^\delta, a_\beta^\gamma] = \delta_{\alpha, \beta\delta-1} a_{\beta\delta-1}^{\delta\gamma} - \delta_{\alpha, \beta\gamma} a_\beta^{\gamma\delta}.$$

Fix $\bar{q} = (q_1, \dots, q_N) \in (\mathbb{C}^\times)^N$ and define the homomorphism $\chi_{\bar{q}} : \mathbb{Z}^N \rightarrow \mathbb{C}^\times$, such that $\chi_{\bar{q}}(\alpha) = \bar{q}^\alpha$, where $\alpha \in \mathbb{Z}^N$ and $\bar{q}^\alpha = q_1^{\alpha_1} \cdots q_N^{\alpha_N}$. The Lie

algebra $\mathfrak{s}(R, \chi_{\bar{q}})$ of formal distributions is a Lie algebra with generators a_m^α , $\alpha \in \mathbb{Z}^N$, $m \in \mathbb{Z}$. Define the generating functions $a^\alpha(z) = \sum_{m \in \mathbb{Z}} a_m^\alpha z^{-m-1}$. From (3.4) we have:

$$[a^\alpha(z), a^\beta(w)] = \sum_{\gamma \in \mathbb{Z}^N} (a_{(\gamma)}^\alpha a^\beta)(w) \delta(z - \chi(\gamma)w),$$

and for modes:

$$\begin{aligned} [a_m^\alpha, a_n^\beta] &= \sum_{\gamma \in \mathbb{Z}^N} \chi(\gamma)^m (a_{(\gamma)}^\alpha a^\beta)_{m+n} \\ &= \bar{q}^{-\alpha m} (T_{-\alpha} a^{\alpha+\beta})_{m+n} - \bar{q}^{\beta m} (a^{\alpha+\beta})_{m+n} \\ &= (\bar{q}^{\alpha n} - \bar{q}^{\beta m}) a_{m+n}^{\alpha+\beta} \end{aligned}$$

This is a vector generalization of sin-algebra, considered in [G-K-L].

5 Non-commutative generalization of Γ -local formal distributions.

In §3 we associated to a Γ -conformal algebra R a Lie algebra $\mathfrak{s}(R, \chi)$ of $\chi(\Gamma)$ -local formal distributions, by fixing a homomorphism $\chi : \Gamma \rightarrow \mathbb{C}^\times$. Conversely, each Lie algebra of $\chi(\Gamma)$ -conformal formal distribution defines a right $\chi(\Gamma)$ -conformal algebra. In this paragraph we will consider more general case, where $\chi(\Gamma)$ is a subgroup of the group of meromorphic transformations of \mathbb{C} . In all examples we will consider $\chi : \Gamma \rightarrow GL_2(\mathbb{C})$. For $\alpha \in \Gamma$ we will denote by $\alpha(z)$ the corresponding under the homomorphism χ transformation from $GL_2(\mathbb{C})$. Consider the generalization of the property (1.2) of the δ -function:

Proposition 5.1 *Let $\alpha(z)$ and $\beta(z)$ be from $GL_2(\mathbb{C})$, then:*

$$\begin{aligned} \alpha'(z) \delta(\alpha(z) - w) &= \delta(z - \alpha^{-1}(w)) \\ \beta'(w) \delta(z - \beta(w)) &= \delta(\beta^{-1}(z) - w) \end{aligned} \tag{5.1}$$

Proof: The group $GL_2(\mathbb{C})$ is generated by the transformations a) $z \mapsto \alpha z$, $\alpha \in \mathbb{C}$; b) $z \mapsto z - \alpha$, $\alpha \in \mathbb{C}$; c) $z \mapsto \frac{1}{z}$. It is sufficient to prove (5.1) for (a),

(b) and (c) transformations. For (a) this is (1.3). To prove (b), let

$$\delta(z - (w - \alpha)) = \sum_{k \in \mathbb{Z}} z^{-k-1} (w - \alpha)^k \quad (5.2a)$$

$$\delta((z + \alpha) - w) = \sum_{k \in \mathbb{Z}} (z + \alpha)^{-k-1} w^k. \quad (5.2b)$$

We define

$$(z + w)^{-\ell-1} = \sum_{k=0}^{\infty} \frac{(k + \ell)!}{k! \ell!} z^{-\ell-k-1} (-\alpha)^k \quad (5.3)$$

for $\ell \geq 0$. Comparing coefficients of $z^{-k-1} w^\ell$ in right-hand sides of (5.2) and (5.3) we get (b). So, the function

$$\delta(z - w + \alpha) = \sum_k z^{-k-1} (w - \alpha)^k = \sum_\ell (z + \alpha)^{-\ell-1} w^\ell \quad (5.4)$$

is well defined. The proof of (c) is similar. \square

Further we will suppose for simplicity that $\Gamma \subset GL_2(\mathbb{C})$. For an N -element subset $S \subset \Gamma$ we shall use the notation

$$(z - w)_S^N = \prod_{\alpha \in S} (z - \alpha(w)).$$

Definition 5.1 *Two formal distributions $a(z)$ and $b(z)$ with coefficients in a Lie algebra \mathfrak{g} are called S -local if in $\mathfrak{g}[[z, z^{-1}, w, w^{-1}]]$ one has:*

$$(z - w)_S^N [a(z), b(w)] = 0.$$

Proposition 5.2 *If $a(z)$ and $b(z)$ are S -local formal distributions, then there exists a unique decomposition*

$$[a(z), b(w)] = \sum_{\alpha \in S} (a_{(\alpha^{-1})} b)(w) \delta(z - \alpha(w)) \quad (5.5)$$

The formal distributions $(a_{(\alpha^{-1})} b)(w)$ are given by the formula:

$$(a_{(\alpha^{-1})} b)(w) = \text{Res}_z \prod_{\substack{\beta \in S \\ \beta \neq \alpha}} \frac{z - \beta(w)}{\alpha(w) - \beta(w)} [a(z), b(w)].$$

Proof is the same as the proof of Proposition 1.1.

For $\alpha \in \Gamma$ introduce the following operator:

$$T_{\alpha^{-1}}a(z) = \alpha'(z)a(\alpha(z)) \quad (5.6)$$

It is clear, that T_α , $\alpha \in \Gamma$, preserve Γ -locality.

For α -products $(a_{(\alpha)}b)(w)$ defined as (5.5) we have all the properties of left Γ -conformal algebras (see Definition 3.1). Conversely, if we have a Γ -conformal algebra R and a homomorphism $\chi : \Gamma \rightarrow GL_2(\mathbb{C})$, we can construct a Lie algebra $\mathfrak{s}(R, \chi)$ of $\chi(\Gamma)$ local formal distributions as follows. Consider a vector space over \mathbb{C} with the basis a_n , $a \in R$, $n \in \mathbb{Z}$, and denote by $\mathfrak{s}(R, \chi)$ the quotient of this space by the \mathbb{C} -space of elements of the form:

$$\begin{aligned} (\lambda a + \beta b)_n - \lambda a_n - \beta b_n, & \quad \lambda, \beta \in \mathbb{C}, \quad a, b \in R \\ (T_\alpha^{-1}a)_n - (\alpha'(z)a(\alpha(z)))_n, & \quad \alpha \in \Gamma \end{aligned} \quad (5.7)$$

where $(\alpha'(z)a(\alpha(z)))_n$ denotes the coefficient of z^{-n-1} in the Fourier series decomposition. Then the formula

$$[a(z), b(w)] = \sum_{\alpha \in \Gamma} (a_{(\alpha^{-1})}b)(w)\delta(z - \alpha(w)) \quad (5.8)$$

gives a Lie algebra structure on $\mathfrak{s}(R, \chi)$ of Γ -local formal distributions $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$.

Remark 5.1 If we let $a(z)T_\alpha = \alpha'(z)a(\alpha(z))$, then we have all properties of right Γ -conformal algebras. We prefer (the more customary) left Γ -modules.

Consider some important examples.

Example 5.1 *Lie algebra of pseudodifferential operators on the circle.*

Let R be the general \mathbb{Z} -conformal algebra $gc_1(\mathbb{Z})$ defined by (4.7). Fix a homomorphism $\chi : \mathbb{Z} \rightarrow GL_2(\mathbb{C})$ defined as $\chi(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$. The corresponding Lie algebra $\mathfrak{s}(\mathbb{Z}, \chi)$ is a Lie algebra with basis a_k^m , $m, k \in \mathbb{Z}$. Let $a^m(z) = \sum_k a_k^m z^{-k-1}$ be the generating function. Then from (5.8) we have:

$$\begin{aligned} [a^m(z), a^n(w)] &= \sum_{s \in \mathbb{Z}} (a_{(s)}^m a^n)(w)\delta(z - w + s) \\ &= (T_{-m}a^{m+n})(w)\delta(z - w - m) - a^{m+n}(w)\delta(z - w + n) \end{aligned} \quad (5.9)$$

To write down explicitly the commutation relations for coefficients of formal distributions $a^m(z)$, consider the Lie algebra $\text{PDiff}(S^1)$ of pseudodifferential operators on the circle (see [Kh-L-R] for more details). This is a Lie algebra of the associative algebra with the basis $x^m \partial^n$, $m, n \in \mathbb{Z}$. The commutation relations are defined from the commutation relations on x and ∂ :

$$\begin{aligned}\partial \circ f(x) &= f(x) \circ \partial + f'(x) \\ \partial^{-1} \circ f(x) &= \sum_{n=0}^{\infty} (-1)^n f^{(n)}(x) \partial^{-n-1},\end{aligned}$$

where $f(x) \in C^\infty(S^1)$. Introduce a new basis of $\text{PDiff}(S^1)$ with generators $x^m D^k$, $m, k \in \mathbb{Z}$, where $D = x\partial$. These generators have more simple commutation relations:

$$[x^m D^k, x^n D^\ell] = x^{m+k} \left((D+n)^k D^\ell - (D+m)^\ell D^k \right) \quad (5.10)$$

For $k < 0$ we will understand $(D+n)^k$ as an expansion by the negative power of D (see (5.3)). Introduce the generating function:

$$a^m(z) = - \sum_{k \in \mathbb{Z}} x^{-m} D^k z^{-k-1}. \quad (5.11)$$

Using the property (5.4) of δ -function, we have

$$\begin{aligned}[a^m(z), a^n(w)] &= \sum_{k, \ell \in \mathbb{Z}} x^{-(m+n)} \left((D-n)^k D^\ell z^{-k-1} w^{-\ell-1} - (D-m)^\ell D^k z^{-k-1} w^{-\ell-1} \right) \\ &= x^{-(m+n)} \left(\sum_{k, \ell \in \mathbb{Z}} D^k D^\ell (z+n)^{-k-1} w^{-\ell-k-1} w^k - D^\ell D^k (w+m)^{-\ell-k-1} (w+m)^k z^{-k-1} \right) \\ &= a^{m+n}(w+m) \delta(z-w-m) - a^{m+n}(w) \delta(z-w-n). \quad (5.12)\end{aligned}$$

Comparing (5.9) and (5.12), we see that the Lie algebra $\mathfrak{s}(\Gamma, \chi)$ is isomorphic to the Lie algebra $\text{PDiff}(S^1)$.

Thus, we have shown, that the Lie algebra of $q - \text{PDiff}(S^1)$ and the Lie algebra of $\text{PDiff}(S^1)$ correspond to the same general \mathbb{Z} -conformal algebra $gc_1(\mathbb{Z})$, but to the different homomorphisms $\mathbb{Z} \rightarrow GL_2(\mathbb{C})$.

More generally, we can consider the general conformal algebra $gc_1(GL_2(\mathbb{C}))$ with the usual action of $GL_2(\mathbb{C})$ on \mathbb{C} . Let $A, B, C \in GL_2(\mathbb{C})$. By (4.7) we have:

$$a_{(C)}^A a^B = \delta_{C, A^{-1}} T_{A^{-1}} a^{AB} - \delta_{C, B} a^{BA}.$$

It is clear, that the general conformal algebra admit reduction to the subalgebra. This reduction is well defined at the level of the Lie algebra $\mathfrak{s}(gl_1(GL(\mathbb{C})))$:

$$[a^A(z), a^B(w)] = A(w)'_w a^{AB}(A(w))\delta(z - A(w)) - a^{BA}(w)\delta(z - B^{-1}(w)). \quad (5.13)$$

As was shown, for the subgroup $H_1 = \left\{ \begin{pmatrix} q^{\frac{m}{2}} & 0 \\ 0 & q^{-\frac{m}{2}} \end{pmatrix}, m \in \mathbb{Z} \right\}$ we get the Lie algebra of q -PDiff(S^1); for the subgroup $H_2 = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, m \in \mathbb{Z} \right\}$ we get the Lie algebra PDiff(S^1). For the subgroup $H_3 \subset GL_2(\mathbb{C})$, generated by two elements $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} q^{\frac{1}{2}} & \\ & q^{-\frac{1}{2}} \end{pmatrix}$ we get the Lie algebra, that contains as subalgebra the Lie algebra of PDiff(S^1) and q -PDiff(S^1). Taking different subgroups H in $GL(2, \mathbb{C})$ (not necessarily discrete) we get a family of infinite-dimensional Lie algebras $\mathfrak{s}(gc_1(H))$.

Example 5.2 Consider the group Γ generated by two elements ϵ and T with relations: $\epsilon^2 = 1$ and $\epsilon T = T^{-1}\epsilon$. Fix two homomorphisms $\varphi_i : \Gamma \rightarrow \text{Aut}(gl_\infty)$, $i = 1, 2$.

$$\varphi_1 : \begin{cases} \epsilon(E_{ij}) = -E_{-j, -i} \\ T(E_{ij}) = E_{i+1, j+1} \end{cases} \quad \text{and} \quad \varphi_2 : \begin{cases} \epsilon(E_{ij}) = E_{-i, -j} \\ T(E_{ij}) = E_{i+1, j+1} \end{cases}$$

In both cases gl_∞ is a free $\mathbb{C}[\Gamma]$ module with the basis $A^m = E_{0, m}$, $m \in \mathbb{Z}_+$. The Γ -products on generators A^m are as follows:

a) For φ_1 we have:

$$\begin{aligned} A_{(\gamma)}^m A^n &= [T^r A^m, A^n] = \delta_{r, -m} T^{-m} A^{m+n} - \delta_{r, n} A^{m+n} \quad \text{for } \gamma = T^r, \\ A_{(\gamma)}^m A^n &= [\epsilon T^r A^m, A^n] = -\delta_{r, 0} T^{-m} A^{n+m} + \delta_{m+n, -r} A^{m+n} \quad \text{for } \gamma = \epsilon T^r. \end{aligned}$$

b) For φ_2 we have:

$$\begin{aligned} A_{(\gamma)}^m A^n &= \delta_{r, -m} T^{-m} A^{m+n} - \delta_{r, n} A^{m+n} \quad \text{for } \gamma = T^r, \\ A_{(\gamma)}^m A^n &= \delta_{r, -m} T^m B^{n-m} - \delta_{r, -n} B^{n-m} \quad \text{for } \gamma = \epsilon T^r \text{ and } n \geq m. \end{aligned}$$

These give us two structures of a Γ -conformal algebra, which we will denote R_1 and R_2 respectively. Consider two homomorphisms $\chi_i : \Gamma \rightarrow$

$GL_2(\mathbb{C})$, $i = 1, 2$:

$$\begin{aligned} \text{a) } \chi_1(\epsilon) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_1(z) = -\frac{1}{z}; \\ \chi_1(T) &= \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad T_1(z) = qz, \\ \text{b) } \chi_2(\epsilon) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon_2(z) = -z; \\ \chi_2(T) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T(z) = z + 1, \end{aligned}$$

We can define four Lie algebras $\mathfrak{s}(R_i, \chi_j)$, $i, j = 1, 2$, of Γ -local formal distributions with generators A_n^m , $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, and the generating function $A^m(z) = \sum_n A_n^m z^{-n-1}$.

(a) $S(R, \chi_1)$:

$$\begin{aligned} [A^m(z), A^n(w)] &= q^m A^{m+n}(q^m w) \delta(z - q^m w) - A^{m+n}(w) \delta(z - q^{-n} w) \\ &\quad - q^m A^{m+n}(q^m w) \delta\left(z + \frac{1}{w}\right) + A^{m+n}(w) \delta\left(z + \frac{q^{m+n}}{w}\right) \end{aligned}$$

For the coefficients A_k^m we have the commutation relations

$$[A_k^m, A_\ell^n] = (q^{-m\ell} - q^{-nk}) A_{k+\ell}^{m+n} - (-1)^k q^{mk} (q^{-n\ell} - q^{nk}) A_{\ell-k}^{m+n}$$

This is the C -series of Lie algebras of quantum torus ([G-K-L]).

(b) $\mathfrak{s}(R_2, \chi_1)$:

$$\begin{aligned} [A^m(z), A^n(z)] &= A^{m+n}(q^m w) \delta(z - q^n w) - A^{m+n}(z - q^{-n} w) \\ &\quad - q^{-m} A^{n-m}(q^{-m} w) \delta\left(z + \frac{q^m}{w}\right) + A^{n-m}(w) \delta\left(z + \frac{q^n}{w}\right), \\ &\quad \text{for } m, n \in \mathbb{Z}, \quad n \geq m \end{aligned}$$

and

$$[A_k^m, A_\ell^n] = (q^{-m\ell} - q^{-k\ell}) A_{k+\ell}^{m+n} - (-1)^k (q^{m\ell} - q^{nk}) A_{\ell-k}^{n-m}.$$

(c) $S(R_1, \chi_2)$:

$$\begin{aligned} [A^m(z), A^n(w)] &= A^{m+n}(w+m)\delta(z-w-m) - A^{m+n}(w)\delta(z-w+n) \\ &\quad - A^{m+n}(w+m)\delta(z+w) + A^{m+n}(w)\delta(z+w-m-n) \end{aligned} \quad (5.14)$$

Consider in $\text{PDiff}(S^1)$ with the basis (5.11) a subalgebra, stable under the automorphism w defined by: $w(x^n D^k) = x^m(m-D)^k$. This is a Lie algebra with the basis $x^m D^k - w(x^m D^k)$. Introduce the generating function of the form:

$$C^m(z) = - \sum_{k \in \mathbb{Z}} (x^{-m} D^k - x^{-m}(-m-D)^k) z^{-k-1} = a^m(z) + a^m(m-z),$$

where $a^m(z)$ are given by (5.12). It is easy to check, that the fields $C^m(z)$ satisfy equation (5.14). We will call this subalgebra a C -series of $\text{PDiff}(S^1)$.

(d) $S(R_2, \chi_2)$ For generating functions we have:

$$\begin{aligned} [A^m(z), A^n(w)] &= A^{m+n}(w+m)\delta(z-w-m) - A^{m+n}\delta(z-w+n) \\ &\quad + A^{n-m}(w-m)\delta(z+w-m) - A^{n-m}\delta(z+w+n) \end{aligned} \quad (5.15)$$

for $m, n \in \mathbb{Z}_+$, and $n \geq m$. This subalgebra is called the B -series of $\text{PDiff}(S^1)$. It can be defined as a subalgebra stable under the second-order automorphism σ defined by $\sigma(x^n D^n) = x^{-m}(-D)^n$. Relations (5.15) are exactly the relations on generating series $B^m(z) = \sum_{k \in \mathbb{Z}} (x^{-m} D^k + \sigma(x^{-m} D^k)) z^{-k-1} = a^m(z) - a^{-m}(-z)$.

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